Globally inhomogeneous "spliced" universes

D. D. Sokolov and A. A. Starobinskii

Moscow University
and Landau Institute of Theoretical Physics, USSR Academy of Sciences, Moscow

(Submitted October 2, 1974)
Astron. Zh. 52, 1041–1048 (September–October 1975)

Topologically "spliced" Friedmann spaces with less than the critical density are considered. The concept of a *G*-domain (for "ghost" images) is introduced, wherein the minimum splicing parameter is shorter than the typical distance between galaxy clusters. The behavior of small adiabatic perturbations of such spaces is investigated, and it is shown that neither galaxy clusters nor galaxies and quasars can form within a certain type of *G*-domain. All perturbation modes are assumed statistically independent and all modes of the same scale are considered equally probable. While the earth definitely is not located in a *G*-domain, these domains could in principle be situated at distances of the same order as the radius of curvature of the universe ($z \approx 1.7$ if $\rho < \rho_c$). Their existence would be a manifestation of the global inhomogeneity of spliced universes, whereby the splicing parameters would vary from point to point. A search for *G*-domains might either establish that our universe is in fact spliced, or at least raise the estimate of the minimum splicing parameter to several tens of megaparsecs.

PACS numbers: 95.30.+m

1. INTRODUCTION

The evidence gathered from astrophysical observations demonstrates that to a high degree of accuracy our universe may be described by a locally homogeneous and isotropic model, which evidently coincides locally with the ordinary Friedmann model. However, the three-dimensional space of this model may, in general, have a complicated topological structure. Such models, in contradistinction to the Friedmann case, have been called "glued-together" or "spliced" universes; accurate definitions of "spliced" spaces have been given in a recent paper by Shvartsman and one of the present authors.[1] In that paper we have shown that the available observational data would be consistent with spliced models of the universe whose topological structure differs from the customary one even on scales much shorter than the distance to the horizon.

A spliced universe is equivalent to an ordinary Friedmann universe with certain periodicity conditions. In a completely homogeneous and isotropic universe these conditions would be satisfied automatically. But the actual universe is homogeneous and isotropic only in the mean. The periodicity conditions therefore lead to the formation of "ghosts" for all objects. This topic has been developed in some detail previously.[1] The implications of a large-scale magnetic field for the detection of "ghosts" have also been considered by one of the present authors.[1]

In this paper we turn to another aspect of the problem: the influence of the splicing of space upon the spectrum of small perturbations in a locally Friedmann model. Perturbations with an arbitrary wave vector, which are admissible for a Friedmann model,[3] cannot arise in a spliced universe. Hence if the global characteristics of spliced universes (the "splicing" or "gluing" parameter and the shape of the inverse domain, as defined previously[1]) vary from point to point, a behavior that one may naturally call the "global inhomogeneity" of a spliced universe (a more accurate definition is given below), then one finds that the distribution of perturbations may become anisotropic and inhomogeneous. As a result the dispersion of the density perturbations and the corresponding distribution of astronomical objects will also vary from point to point.

We would emphasize that this variation in the dispersion of the density perturbations will occur if one adopts the hypothesis of an equipartition of perturbations over modes with the same characteristic scale, as well as the statistical independence of different modes, a hypothesis that leads to the homogeneous pattern of perturbations in an ordinary nonspliced model. With this statistical hypothesis, the rms combined perturbation is the same at every point of a Friedmann space. In the case of a spliced universe we shall find that the rms perturbation is inhomogeneous, depending on the coordinates. We thereby have a differing probability for the formation of clusters of galaxies at different points of space. The topological structure of spliced universe therefore may be investigated not only by searching for "ghosts" but also by investigating singularities in the distribution of galaxy clusters over the sky.

We shall confine attention in this paper to the case of a universe with much less than the critical density, and to types of splicing such that the shape of the inverse domain remains constant, even though the splicing parameters vary from point to point. Under these conditions the effects of interest to us are most clearly and simply expressed, and furthermore, various arguments indicate that the density of the universe is indeed much less than the critical value.

As will be shown below, the singularities in the distribution of galaxy clusters under these circumstances will take the form of dark and light spots or bands. The regions of space corresponding to these singularities will be called "G-domains" (for "ghosts"). Their geometrical properties are discussed in Sec. 2; the structure of the perturbations within them, in Sec. 3.

If they do exist, G-domains could be detected comparatively easily; since a large number of contiguous ghosts of the inverse image of a G-domain may be observed simultaneously, its apparent size will be considerably larger than the apparent distances between galaxy clusters located at the same distance. The following relation holds for $\Omega = \rho/\rho_c \ll 1$:

$$\sin\frac{\varphi}{2} = \frac{1}{Z+1}, \quad (1)$$

where φ is the angle subtended by the diameter of a G-domain and Z is the red shift of the closest point of its boundary.[1)]

Evidently the earth is not located in a G-domain. This circumstance supports in a different way the estimate $l \gtrsim 10$ Mpc for the minimum splicing parameter, as obtained from a search for ghosts.[1] If G-domains should remain undetected out to sufficiently large values of Z, then lower limits amounting to several tens of megaparsecs would be placed on the minimum splicing parameter for a series of models (the best conceivable lower limit would be 100 Mpc).

2. GEOMETRY OF G-DOMAINS

If the density of the universe is lower than the critical density, then space will coincide locally with Lobachevskii space. Any spliced universe of the type we are considering can be obtained in the following manner. From Lobachevskii space we isolate a piece bounded by some set of pairs of nonintersecting planes, and we identify planes in pairwise correspondence without rotation. If certain conditions on this set of paired planes are satisfied, we will obtain a spliced universe. The specific nature of these conditions will play no role for the discussion to follow.

We now introduce the concept of a G-domain for each pair of nonintersecting planes. Let $\Lambda = 10$ Mpc be the characteristic distance between clusters of galaxies. The inverse image of a G-domain is a region of spliced space in which the distance between the paired planes is less than Λ (one will recall that in Lobachevskii space planes can never be equidistant,[2)] that is, located at a constant distance apart). A G-domain is defined as a region in Lobachevskii space consisting of its inverse region together with all the ghosts of the inverse; the G-domain constitutes the observed image of its own inverse.

A spliced universe will be called globally inhomogeneous if its maximum and mimumum splicing parameters L(B), l(B), as well as the shape of the inverse domain H(B) or the position of the point B within H(B), vary from place to place,[3)] that is, if they depend on the choice of the point B. A further characterization of the global inhomogeneity of a spliced universe (in terms of eigenfunctions) will be given in Sec. 3.

One finds that every spliced hyperbolic universe is globally inhomogeneous. On the other hand, a homogeneous and isotropic universe with flat three-dimensional space (the "flat Friedmann model") can be so spliced that it will be globally homogeneous. An example of such a universe is a Friedmann model with three-dimensional space in the form of a plane three-torus.

Let us return to a hyperbolic universe (with $\rho < \rho_c$). We need only consider the G-domains that arise for two qualitatively distinct pairs of nonintersecting planes: parallel and divergent. In fact, one can verify that the G-domains generated by different pairs of planes will affect each other quite independently.

The space metric of the universe (a cylindrical "horn") obtained through the identification of two parallel planes will have the form

$$dl^2 = R^2[dx^2 + e^{-2x}(dy^2 + dz^2)], \quad (2)$$

where R is the radius of curvature; the correspondence relations are

$$y = y + ma, \quad x = x, \quad z = z; \quad m = 0, 1, 2 \ldots . \quad (3)$$

The metric of Lobachevskii space written in the form (2) may be reduced to the ordinary spherical form

$$dl^2 = R^2[dr^2 + \text{sh}^2 r(d\theta^2 + \sin^2\theta d\varphi^2)] \quad (4)$$

by means of the transformation

$$x = -\ln(\text{ch}\, r - \text{sh}\, r\cos\theta), \qquad y = \frac{\sin\theta\cos\varphi}{\text{cth}\, r - \cos\theta}, \quad z = \frac{\sin\theta\sin\varphi}{\text{cth}\, r - \cos\theta}. \quad (5)$$

The boundary of the G-domain is described by the equation

$$Rae^{-x} = \Lambda. \quad (6)$$

Without loss of generality we may regard the observer as located at the point $r \equiv (x, y, z) = 0$. Then for $\rho \ll \rho_c$ the red shift of objects located at the x boundary of the G-domain will be expressed by the equation $Z = e^x - 1$.

Such a G-domain represents the interior of an orisphere (the generalization to Lobachevskii space of a sphere of infinite radius), and is called an orispherical G-domain.

The metric of the universe obtained through the identification of two divergent planes ("bowls") will have the form

$$dl^2 = R^2[dx^2 + \text{ch}^2 x(dy^2 + \text{ch}^2 y dz^2)]. \quad (7)$$

The correspondence relations are

$$z = z + ma, \quad x = x, \quad y = y; \quad m = 0, 1, 2 \ldots . \quad (8)$$

The metric (7) may be reduced to the spherical form (4) by means of the transformation

$$\text{sh}\, x = \text{sh}\, r\cos\theta, \quad \text{sh}\, y = \frac{\text{sh}\, r\sin\theta\cos\varphi}{\sqrt{1 + \text{sh}^2 r\cos^2\theta}}, \qquad \text{th}\, z = \text{th}\, r\sin\theta\sin\varphi. \quad (9)$$

In this universe a G-domain will exist if the condition $Ra \leq \Lambda$ is satisfied. The equation of its boundary will be

$$Ra\,\text{ch}\, x\,\text{ch}\, y = \Lambda. \quad (10)$$

Such a G-domain represents the interior of an equidistant "barrel" (a set of points located at a fixed distance from a given line); it is called an equidistant G-domain.

We shall analyze in detail the case of a "toroidal horn," which has the metric (2) but with the correspondence relations

$$y = y + ma, \quad z = z + nb, \quad x = x; \quad m, n = 0, 1, 2 \ldots . \quad (11)$$

We shall further assume that $a \approx b$. The corresponding G-domain of the horn is the superposition of two orispherical G-domains. The effects to be considered are most characteristic for a G-domain of this type. For such a splice the maximum diameter D of the universe is infinite, while the minimum diameter d = 0 (see the previous paper[1] for definitions of D and d).

The inverse of the G-domain of a horn has the finite volume[4)] $V \approx \Lambda^2 R \approx 600\ \Lambda^3$. Other types of G-domains may have inverses with either finite or infinite volumes.

It may readily be verified that if a horn can be proved to have no G-domain out to some red shift Z_1, then the following estimate for the minimum splicing parameter will be applicable for the earth:

$$l \geqslant (Z_1+1)\Lambda. \tag{12}$$

Galaxy formation occurred no earlier[6] than Z = 5-10 (observations are in fact possible only for still smaller Z). Hence the method outlined above can yield estimates for l of no more than several tens of megaparsecs.

One can readily show that the apparent angular diameter of a G-domain is determined by Eq. (1) and depends very weakly on the form of the G-domain.

3. STRUCTURE OF PERTURBATIONS IN G-DOMAINS

In this section we shall investigate the singularities in the spectrum of small density perturbations in G-domains, and the observational manifestations of such singularities. The theory of perturbations and its relationship to the eigenfunction spectrum of the covariant Laplace operator was developed in Lifshits' fundamental paper.[7] Zel'dovich[3] has pointed out that splicing will alter the eigenfunction spectrum and may influence the growth of perturbations; see also the discussion by Paal.[8]

In examining the theory of perturbations in a spliced universe we shall adopt the same hypothesis regarding the character of the initial spectrum of the perturbations as in the ordinary Friedmann model. All modes will be assumed statistically independent, and at a given wavelength (that is, for perturbations of given mass) all modes will be considered equally probable. In a globally homogeneous universe these assumptions imply that the amplitude of the perturbations will, on the average, be the same at all points of space at any given time. On the average the amplitude of the perturbations in a globally inhomogeneous universe will be approximately constant outside G-domains, but will decay rapidly inside horn-type G-domains (orispherical G-domains) or grow inside bowl-type G-domains (equidistant G-domains). The physical significance of our assumptions is that the factor responsible for the initial perturbation spectrum will be independent of the splicing. At present we do not know what factors might have generated the primordial spectrum of perturbations. Thus other hypotheses regarding that spectrum are also logically possible, whereby special types of objects would be formed in G-domains (see below).

Every small perturbation of the Friedmann model (including spliced models) can be expanded in eigenfunctions of the three-dimensional covariant Laplace operator with time-dependent coefficients. We shall not consider the time dependence of the perturbations, as it will not be altered by splicing and hence is already established by Lifshits' analysis.[7] Moreover, we shall be interested only in density perturbations, which are described by a scalar function ψ.

The following general method may be applied to obtain a system of eigenfunctions in a spliced universe. We take a complete system of eigenfunctions of the same operator in the corresponding ordinary Friedmann model, and select those and only those linearly independent functions that are invariant with respect to the correspondence relations (these functions may be linear combinations of original functions). Evidently in this event we can always find certain eigenfunctions for the ordinary Friedmann model that do not belong to the complete system of eigenfunctions for the spliced universe. Of course we do not know in advance whether splicing is present, so that we expand the perturbations with respect to the eigenfunctions of the Friedmann model. The "missing" modes will then be represented by gaps. In principle, the manner in which these gaps alternate would enable us to determine the type of splicing. For example, in the case of a plane three-torus, which is a globally homogeneous space, the perturbation spectrum will be discrete, with wave vectors of the form

$$\left(\frac{2\pi m}{a}, \frac{2\pi n}{b}, \frac{2\pi p}{c}\right),$$

where a, b, c are the diameters of the torus. In a toroidal universe the perturbations will have a discrete set of wavelengths, but this effect will be perceived only if the wavelength is comparable to the dimensions of the torus.

In globally inhomogeneous universes this discrete behavior will be strongly apparent in G-domains. Let us consider the eigenfunctions in a toroidal horn with the metric (2). The restricted eigenfunctions of the Laplace operator, satisfying the equation

$$\Delta\psi + (k^2+1)\psi = 0 \tag{13}$$

and normalized by $\delta(k-k')\delta_{mm'}\delta_{nn'}$, have the form

$$\psi_{kmn}(\mathbf{r}) = \sqrt{\frac{2k\,\mathrm{sh}\,\pi k}{\pi^2}\cdot\frac{(2-\delta_{m0})(2-\delta_{n0})}{ab}} \times e^x K_{ik}(Qe^x)\begin{pmatrix}\sin\frac{2\pi m}{a}y\\ \cos\frac{2\pi m}{a}y\end{pmatrix}\begin{pmatrix}\sin\frac{2\pi n}{b}z\\ \cos\frac{2\pi n}{b}z\end{pmatrix}, \tag{14}$$

where m, n = 0, 1, 2, ...; $m^2+n^2 \neq 0$; $Q = 2\pi[(m^2/a^2) + (n^2/b^2)]^{1/2} > 0$; and $K_\nu(\tau)$ is the MacDonald function. The quantity k is related to the wavelength Λ of the perturbation by $\Lambda = 2\pi R/k$, where R(t) is the radius of curvature. In the absence of splicing, the system of eigenfunctions for the metric (2) takes the form given by Vilenkin and Smorodinskii.[9]

As $x \to \infty$ (properly speaking, for $Qe^x \gg k$), the functions (14) approach $\exp(-Qe^x)$ asymptotically. Since all the restricted eigenfunctions decay in a G-domain $\{x > \ln[(R/\Lambda)\max(a, b)]\}$, clusters of galaxies will not be formed there. According to current views,[6] individual galaxies arise from clusters, so that separate galaxies will not be formed either in a G-domain of this type. Thus a horn-type G-domain should manifest itself observationally as a dark spot in the distribution of galaxies. The dimensions of the spot are given by Eq. (1).

This effect results directly from the splicing. In unspliced Lobachevskii space, m and n can be selected arbitrarily close to zero, so that functions will always exist depending on y and z and decreasing with x as slowly as desired. In a spliced universe m and n must be at least unity.

The spot effect admits of the following geometrical interpretation. Consider a beam of test particles whose average motion is along the x direction. By examining the grid of geodetic curves on a toroidal horn, one can easily show that every particle with a velocity vector not parallel to the x axis will sooner or later change the sign of the x coordinate of its velocity and will be "reflected" from the horn (the smaller the slope of the velocity vector relative to the x axis, the later the change in sign will occur). Arnol'd[10] has given an analysis of this phenomenon in his book.

Under the hypothesis we have formulated above regarding the character of the initial perturbation spectrum, the space distribution of perturbations with wavelength $\Lambda = 2\pi R/k$ may be characterized by the quantity

$$S_k(\mathbf{r}) = \sum_{\substack{m,n \\ k=\mathrm{const}}} |\psi_{kmn}(\mathbf{r})|^2, \tag{15}$$

which is readily expressed in terms of the Green's function $G_k(\mathbf{r}, \mathbf{r}')$ of Eq. (13) for coincident values of the arguments:

$$S_k(\mathbf{r}) = -\frac{2k}{\pi}\,\mathrm{Im}\, G_k(\mathbf{r}, \mathbf{r}). \tag{16}$$

In the case of a globally homogeneous universe, $G_k(\mathbf{r}, \mathbf{r}') = G_k(\mathbf{r}-\mathbf{r}')$, and then $S_k(\mathbf{r}) \equiv \mathrm{const} = k^2/2\pi^2$. The variability of $S_k(\mathbf{r})$ describes the global inhomogeneity of the universe resulting from splicing. For the case of splicing that we have considered [Eqs. (11)], we obtain

$$S_k(\mathbf{r}) \approx \frac{k^2}{2\pi^2} \text{ for } e^x Q_{\min} \ll 1 \tag{17}$$

and

$$S_k(\mathbf{r}) \approx \frac{2k\,\mathrm{sh}\,\pi k}{\pi a b} Q_{\min}^{-1} \exp(-2Q_{\min}e^x + x)$$

if $e^x Q_{\min} \gg k$, that is, if $x \gg \ln[(R/\Lambda)\max(a, b)]$; here $Q_{\min} = 2\pi \min(a^{-1}, b^{-1})$. The quantity $S_k(\mathbf{r})$ will decay rapidly inside a G-domain as $x \to \infty$, which further emphasizes the very low probability that perturbations will develop in a given G-domain as compared to other regions of space.

In a toroidal horn, eigenfunctions depending only on x will also be possible; with Q = 0 they will have the form

$$\psi_k(x) = \frac{1}{\sqrt{\pi a b}} e^x \begin{pmatrix} \sin kx \\ \cos kx \end{pmatrix}. \tag{18}$$

These functions are normalized by the δ function, but are not bounded as $x \to \infty$; hence they are not contained in the system of functions (14). The perturbations proportional to them cannot be considered small, but from a set of these functions one can form a small perturbation depending only on x. In our statistical hypothesis the probability of so small a perturbation would be negligible, as it would require a definite correlation between an infinite number of modes of differing wavelength.

If we consider the case of a special excitation of the appropriate modes, then the formation of a special type of object whose structure depends only on x can be achieved in a G-domain. During the nonlinear phase such an object cannot be regarded as local; its evolution will proceed differently from the evolution of an ordinary cluster of galaxies. In fact, in the conventional model such a perturbation would correspond to a change in density throughout the region confined between two orispheres. The apparent dimensions of such an object will again be determined by Eq. (1). This example shows that even if the statistical hypothesis adopted above is not fulfilled, the presence of splicing might nonetheless be detected observationally.

Now let us consider a bowl-type G-domain (an equidistant G-domain) with the metric (7) and the splice (8). A singular role will here be played by modes independent of z which are bounded, unlike the preceding case. The eigenfunctions will now have the form

$$\psi_{kpm}(\mathbf{r}) = \sqrt{\frac{kp\,\mathrm{sh}\,\pi k\,\mathrm{sh}\,\pi p}{(\mathrm{ch}\,2\pi k + \mathrm{ch}\,2\pi p)(\mathrm{ch}\,2\pi p + \mathrm{ch}\,2\pi q)} \frac{2-\delta_{m0}}{a}} \times \frac{1}{\sqrt{\mathrm{ch}\,y}\,\mathrm{ch}\,x} P^{-ip}_{-1/2+iq}(\mathrm{th}\,y) P^{-ik}_{-1/2+ip}(\mathrm{th}\,x) \begin{pmatrix} \sin\frac{2\pi m}{a} z \\ \cos\frac{2\pi m}{a} z \end{pmatrix}, \tag{19}$$

where $q = 2\pi m/a$, $m = 0, 1, 2, \ldots$, and the $P_\mu^\nu(\tau)$ are the associated Legendre functions. By definition of a G-domain, $\Lambda > Ra$ ($ka < 2\pi$); furthermore, the wavelength of the perturbations of interest to us is shorter than the horizon, so that $k \ll 1$.

We construct the quantity $S_k(\mathbf{r})$ [Eq. (15)] by means of the system of functions (19). Outside the G-domain, where $ka \gg 1$ (that is, for $Ra \cosh x \cosh y \gg \Lambda$), we find that $S_k(\mathbf{r}) \approx k^2/2\pi^2$, as expected. Inside the G-domain, where $ka \cosh x \cosh y \ll 1$, the quantity $S_k(\mathbf{r}) \approx (k/2\pi a)[(\cosh x)\cdot\cosh y]^{-1} \gg k^2/2\pi^2$, and this value results primarily from modes with m = 0 (the contribution from modes with $m \neq 0$ is exponentially small). Clearly there is a high probability that perturbations independent of z will develop inside the G-domain. Galaxies and quasars can subsequently form from these perturbations. Hence a bowl-type G-domain should appear observationally as a spot whose angular size is given approximately by Eq. (1). Inside the G-domain, galaxies should be accumulated not in clusters but in narrow parallel bands with a characteristic width of order Λ and a length equal to the diameter of the G-domain (the distance between the bands should also be of order Λ).

In conclusion we would point out that two cases have been studied here in detail, representing typical examples of the influence of splicing on the structure of perturbations; G-domains of these types would most readily be detectable observationally, or their absence demonstrated.

The variation in the characteristic distance Λ between clusters will serve to spread out the boundary of a G-domain somewhat. But Eqs. (6) and (10) imply that the boundary of a G-domain has only the weak Λ dependence $x \propto \ln \Lambda$; the red shift corresponding to this value of x will conform to the law $Z \propto \Lambda$.

The authors are indebted to Ya. B. Zel'dovich for many conversations that provided the stimulus for this investigation, as well as to N. V. Efimov, A. B. Ivanov, A. V. Kovalenko, É. G. Poznyak, A. A. Ruzmaikin, V. F. Shvartsman, and A. I. Shniel'man for most helpful discussions.

1) To the observer this point will appear to be centered in the G-domain.
2) Elementary information on Lobachevskii geometry that we shall need may be found in standard monographs.[4,5]
3) All the quantities introduced here have been defined previously.[1] In particular, at distances $r > L$ from the observer there exists no inverse domain; at distances $r < l$ there exist no ghosts.
4) The Hubble constant has here been taken equal to 55 km·sec^{-1}·Mpc^{-1}.

[1] D. D. Sokolov and V. F. Shvartsman, Zh. Eksp. Teor. Fiz., 66, 412 (1974) [Sov. Phys. – JETP, 39, 196 (1975)].
[2] D. D. Sokolov, Pis'ma Astron. Zh., 1, 3 (1975) [Sov. Astron. Lett., 1, 113 (1975)].
[3] Ya. B. Zel'dovich, Comments Astrophys. Space Phys., 5, 169 (1973).
[4] N. V. Efimov, Higher Geometry [in Russian], Nauka, Moscow (1971).
[5] B. A. Rozenfel'd, Non-Euclidean Space [in Russian], Nauka, Moscow (1969).
[6] Ya. B. Zel'dovich and I. D. Novikov, Relativistic Astrophysics, Vol. 2, Univ. Chicago Press (in press).
[7] E. M. Lifshits, Zh. Eksp. Teor. Fiz., 16, 587 (1946).
[8] G. Paal, Acta Phys. Acad. Sci. Hung., 30, 51 (1971).
[9] N. Ya. Vilenkin and Ya. A. Smorodinskii, Zh. Eksp. Teor. Fiz., 46, 1793 (1964) [Sov. Phys. – JETP, 19, 1209 (1964)].
[10] V. I. Arnol'd, Mathematical Methods of Classical Mechanics [in Russian], Nauka, Moscow (1974).

Translated by R. B. Rodman

The second strong-focusing region formed by a non-Schwarzschild gravitational lens

L. Kh. Ingel' and N. R. Rubakha

(Submitted July 9, 1974)
Astron. Zh. 52, 1049–1054 (September–October 1975)

Non-Schwarzschild gravitational lenses will produce a second, two-dimensional region where strong focusing can occur, but an allowance for diffraction spreading considerably reduces the size of this region. Various possibilities for observing gravitational-focusing effects (especially for neutrinos and gravitational waves) are discussed.

PACS numbers: 04.40.+c

We shall consider the focusing of electromagnetic waves1) in the field of a gravitational lens. A cylindrical coordinate system will be adopted with its z axis passing through the source and the center of the lens, which we shall take as the origin (see Fig. 1). The lens will be regarded as symmetric about the z axis. It is apparent from simple geometrical considerations that the trajectory of a ray with impact parameter p will, after passing through the lens, be described[1–4] by the equation

$$\xi=\frac{z}{L}\left[p-L\Theta(p)\right], \qquad (1)$$

where ξ is the distance from the z axis, $L = lz/(l + z)$, l is the distance between the source and the lens, and $\Theta(p)$ represents the angle by which the lens deflects the ray.

For the coefficient K by which the gravitational-lens effect enhances the observed luminosity of the source, we readily obtain[3,4] the expression

$$K=\left(\frac{l+z}{l}\right)^2\left|\frac{p}{\xi}\frac{dp}{d\xi}\right|. \qquad (2)$$

With Eq. (1) we have

$$K=\frac{1}{\left|1-L\dfrac{\Theta(p)}{p}\right|\left|1-L\dfrac{d\Theta}{dp}\right|}. \qquad (3)$$

Equation (2) shows that the lens forms a caustic curve ($K = \infty$) for $\xi = 0$ (that is, on the z axis). In analyses of gravitational lenses one generally considers the region of strong focusing effect around this caustic. We have pointed out in a recent paper,[4] however, that non-Schwarzschild gravitational lenses (which may be represented, for example, by galaxies for electromagnetic waves and stars for gravitational waves and neutrinos) can give rise to other caustics as well, if the derivative $d\xi/dp$ vanishes. We have introduced the term "caustics of the second type" for these two-dimensional caustics. They are surfaces of revolution about the z axis (Fig. 1). "New" regions of strong focusing effect would be associated with these caustics. A geometrical-optics treatment[4] yields the following result: The volume of the second strong-effect region is of the same order as the volume of the region known previously.2)

The two-dimensional character of caustics of the second type is important. If the lens or the source is moving transverse to the z axis, a two-dimensional caustic will cover far more space than the first strong-effect region.

Several matters are of interest in this connection.

1. In many investigations (see, for example, papers by several authors[1,2,5–8] and the references cited there), the probability of an observable manifestation of the gravitational-lens effect is considered. But the existence of caustics of the second type is not recognized in any of these analyses. Has there not accordingly been a substantial underestimate of the probability of an observable gravita-