ROTATIONAL INSTABILITY OF A GAS DISC

A. M. Fridman

Nuclear Physics Institute, Novosibirsk

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The gravitational instability of a rotating gas disc (p = 0, where p is the gas pressure) is studied by means of an energy principle. The results are applicable to a wide class of perturbation functions. Instability of the disc relative to highly localized radial displacements has been detected.

1. The problem of the stability of flat (rotating) distributions of matter is of interest in the theory of the evolution of galactic systems. In recent years this problem has been investigated by a number of authors (for example, see [1, 2]).

The main difference between investigation of the stability of a rotating gravitating disc and the known characteristics of stability of other bodies of revolution that are astronomical objects (sphere, ellipsoid, studies of which were initiated by Lyapunov, and cylinder [4]) is a nonstandard form of Poisson's equation. In the general case, an attempt to determine the eigenvalues of the system of equations leads to a very complicated integro-differential equation. The feasibility of the reduction of the initial system to a differential equation is associated with a fairly strict restriction on the class of perturbed functions.

Thus, an instability criterion for a rotating gravitating disc was suggested in [1] on the basis of a quasi-classical approximation, i.e., for sufficiently localized displacements. It was shown that instability results in the disc acquiring a spiral structure when the displacements depend on the angle φ.

In [2], where the problem of the stability of the disc was solved by the determination of the eigenvalues of the system of equations (1)-(4), the perturbed functions were chosen of the form $\sim \exp[i(kr + \omega t + m\varphi)]$, which, generally speaking, is without any foundation for an inhomogeneous finite system.

By contrast to [1], where the mathematical problem was reduced to the determination of the eigenvalues of a system of differential equations in the quasi-classical approximation, i.e., for sufficiently localized displacements, and the other preceding papers (see review [3] and the papers cited there), whose authors, following Jeans, considered periodic perturbations leading to the breakup of the system into individual condensations (in other words, to star formation), the present paper uses a variational principle which allows us to draw conclusions concerning the stability of a disc relative to a wide class of perturbed functions. As an example, it is demonstrated that a disc is unstable relative to displacements of the form $\exp(-x^2/\Delta^2)$, which lead to the escape of matter from the edge of the disc.

A physical analog of this type of instability is the rotational instability of a rotating spherical star which causes the ejection of matter from the equator [3].

2. The stability of a rotating cold gas disc (p = 0, where p is the gas pressure) is investigated in the present paper. The stability problem is reduced to the investigation of the time development of small oscillations about the equilibrium state. For low-amplitude oscillations we can make use of the linearized equations of motion. Let σ, v, and ψ represent small deviations of the density, velocity, and potential from their equilibrium values $\sigma^{(0)}$, $v^{(0)}$, and $\psi^{(0)}$. All variable quantities will be assumed to depend only on the coordinate r. We will assume that the perturbations are of the form $A(r, t) = A(r) \exp(-i\omega t + im\varphi)$, where ω is the frequency of the perturbations.

The linearized hydrodynamic equations in cylindrical coordinates for the cases m = 0 can be written in the form[1]

[1]The variational principle used below is applicable when m = 0. For higher modes ($m \neq 0$) the system of Eqs.(1)-(4) is found to be nonself-adjoint.

$$\frac{\partial v_r}{\partial r} + \left(\ln' \sigma^{(0)} + \frac{1}{r}\right) v_r - i\omega \frac{\sigma}{\sigma^{(0)}} = 0, \tag{1}$$

$$i\omega v_r + \frac{2v_\varphi^{(0)}}{r} v_\varphi = \frac{\partial \psi}{\partial r}, \tag{2}$$

$$2\frac{\partial v_\varphi^{(0)}}{\partial r} v_r - i\omega v_\varphi = 0, \tag{3}$$

$$\psi = k \int_{-\pi}^{\pi} \int_{0}^{\alpha} \frac{\sigma(r') r' \, dr' \, d\theta}{\sqrt{r^2 + r'^2 - 2rr' \cos\theta}}. \tag{4}$$

Here k is the gravitational constant, α is the radius of the disc. Equation (1) is the continuity equation, (2) and (3) are the Euler equations of motion, while (4) describes the perturbed gravitational potential of the disc.

We will study Eqs.(1)-(4) with the help of an energy principle.

Let us write Eqs.(2) and (3) in the form

$$\frac{\partial^2 \xi_r}{\partial t^2} = -\frac{\partial \psi}{\partial r} + \frac{2v_\varphi^{(0)}}{r} \frac{\partial \xi_\varphi}{\partial t}, \tag{2a}$$

$$\frac{\partial^2 \xi_\varphi}{\partial t^2} = -2\frac{\partial v_\varphi^{(0)}}{\partial r} \frac{\partial \xi_r}{\partial t}. \tag{3a}$$

Here we have carried out the following change of variables: $v_r = \partial\xi_r/\partial t$, $v_\varphi = \partial\xi_\varphi/\partial t$, where ξ_r, ξ_φ are arbitrary displacements along the coordinates r and φ.

Let us add the last two equations after multiplying the first by ξ_r and the second by ξ_φ. We thus obtain an expression for the variation of the potential energy of small oscillations:

$$\delta W = -\frac{1}{2} \int_{s_0} \sigma^{(0)} \left(\xi_r \frac{\partial^2 \xi_r}{\partial t^2} + \xi_\varphi \frac{\partial^2 \xi_\varphi}{\partial t^2} \right) ds = -\pi \int_0^\alpha r \sigma^{(0)} \xi_r$$

$$\times \left\{ \frac{\partial}{\partial r} \int_{-\pi}^{\pi} \int_0^\alpha \frac{k \frac{1}{r'} \frac{\partial}{\partial r'} (r' \xi_{r'} \sigma_0)}{\sqrt{r^2 + r'^2 - 2rr' \cos\theta}} dr' \, d\theta \right.$$

$$\left. + 4\xi_r r \frac{\partial v_\varphi^{(0)}}{\partial r} \frac{\partial}{\partial r} \left(\frac{v_\varphi^{(0)}}{r} \right) \right\} dr. \tag{5}$$

In deriving formula (5) we have used expression (4), after substituting the density $\sigma(r')$ into it from Eq.(1).

Let us write (5) in the form

$$\delta W = -\frac{1}{2} \int_{s_0} F(\xi_r) \xi_r \, ds, \tag{5a}$$

where $F(\xi_r)$ is the force acting on a surface element of the disc in the radial direction. When $\delta W > 0$, the direction of the force is opposite to the direction of displacement and the disc is stable. When $\delta W < 0$ the disc is unstable.

The equation of the small oscillations (5) can be obtained from the variational principle $\delta\{\int L dt\} = 0$, where L is the Lagrangian, equal to the difference of the kinetic energy

$$T = \frac{1}{2} \int \sigma^{(0)} \left(\frac{\partial \xi_r}{\partial t} \right)^2 ds \tag{6}$$

and the potential energy W. This follows directly from the self-adjointness of Eq.(5) because after some transformations expression (5) can be reduced to a form symmetrical with respect to ξ and η:[2]

$$\delta W = -\pi \int_0^\alpha \left\{ k\sigma^{(0)}(r) \int_0^\alpha \left[\eta(r) \xi(r') \sigma^{(0)}(r') rr' \right. \right.$$

$$\left. \times \frac{\partial^2}{\partial r \partial r'} \int_{-\pi}^{\pi} \frac{d\theta}{\sqrt{r^2 + r' - 2rr' \cos\theta}} \right]$$

$$\left. + 4\sigma^{(0)}(r) \eta(r) \xi(r) \frac{\partial v_\varphi^{(0)}}{\partial r} r^2 \frac{\partial}{\partial r} \left(\frac{v_\varphi^{(0)}}{r} \right) \right\} dr, \tag{7}$$

where η is an auxiliary quantity (displacement) satisfying the same boundary conditions as ξ [$\xi(\alpha) = 0$].

In accordance with observational data on the distributions of density and angular velocities in flat galactic subsystems [5-6], we will consider two particular cases:

$$1) \quad \sigma^{(0)}(r) = \begin{cases} \frac{c_1}{r} \text{ when } r > \varepsilon > 0, \\ \sigma^{(0)} \text{ when } 0 \leqslant r \leqslant \varepsilon, \end{cases} \tag{8}$$

$$2) \quad \Omega^{(0)}(r) = \begin{cases} \frac{c_2}{r} \text{ when } r > \varepsilon > 0, \\ \Omega^{(0)} \text{ when } 0 \leqslant r \leqslant \varepsilon, \end{cases} \tag{9}$$

where $\Omega^{(0)}(r)$ is the angular velocity of rotation of the disc and $c_{1,2}$ = const. Let us take the displacement to be of the form

$$\xi_r = \exp(-\mu x^2), \tag{10}$$

where $\mu = 1/\Delta^2$, Δ being the rms displacement, and $x^2 = (r - r_0)^2$. With this choice of displacement,

[2]Symmetry with respect to r and r' is satisfied automatically.

the largest contribution to expression (5) comes from the vicinity of the point r_0, so that

$$\int_{-\pi}^{\pi} \frac{d\theta}{\sqrt{r^2 - r'^2 - 2rr' \cos \theta}}$$

$$\sim \int_{-\theta_0}^{\theta_0} \frac{d\theta}{\sqrt{(r-r')^2 + rr'\theta}} \sim \ln \frac{2r_0^2}{(r-r')^2}. \tag{11}$$

Expansion (11) is valid for

$$r - r' \sim \Delta \ll r_0, \tag{12}$$

i.e., according to (8), in the region where $\sigma^{(0)}(r) \sim 1/r$.

The choice of displacement in the form (10) is governed by the ease with which integral (5) can be evaluated and it, like any other arbitrary form of the displacement, allows us to establish the sufficiency criterion for the instability of the system.

The system of Eqs.(1)-(4) in the zero-order approximation gives the relationship between the unperturbed values of the velocity and density:

$$\frac{(v_\varphi^{(0)})^2}{r} = k \frac{\partial}{\partial r} \int_{s_0} \frac{\sigma^{(0)}(r')\, ds'}{\sqrt{r^2 + r'^2 - 2rr' \cos \theta}}. \tag{13}$$

Taking this equation into account, as well as (8), (10), and (11), we find the following expression for the potential energy of the system after exponentially small terms have been neglected:[3]

$$\delta W = -K\pi \left(\frac{1}{4} \ln 2\mu r_0^2 + 4 \ln \frac{\alpha}{r_0} \right), \tag{14}$$

where K is a constant with dimensions of energy and is a combination of c_1, α, and k. Using condition (12), we find that $\delta W < 0$.

In the case when the angular velocity of rotation $\Omega^{(0)}(r)$ decreases with increasing distance from the center of the disc (9), the potential energy with small exponential terms neglected takes the form[4]

$$\delta W = -\frac{3K' \pi^{3/2}}{8\mu r_0^2}, \tag{15}$$

where K' is a constant with dimensions of energy.

In order of magnitude, expression (15) is much less than the analogous expression (14); this is explained by the fact that in the case $\Omega^{(0)} \sim 1/r$ the term responsible for rotation in formula (5) becomes zero. Therefore, the instability increment calculated from the formula

$$\omega^2 = \frac{\delta W}{\int_{s_0} \sigma^{(0)} \xi_r^2\, ds} \tag{16}$$

is much higher in case (8) than in case (9) and is given by

$$\omega^2 \sim -\frac{3K_1}{4\sqrt{2}\,\alpha}, \tag{17}$$

where K_1 is a constant.

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[3] In order for us to be able to make use of the table of integrals [7], it is convenient to carry out the following transformation:

$$\ln \frac{r_0^2}{(x-x')^2} = \ln r_0^2 - \ln (x^2 - x'^2)^2 + \ln (x + x')^2,$$

and then to make use of the formula

$$\ln (x^2 - x'^2)^2 = 2 \ln (x^2 - x'^2)\theta(x^2 - x'^2) + 2 \ln (x'^2 - x^2)\,\theta(x'^2 - x^2),$$

where

$$\theta(x^2 - x'^2) = \begin{cases} 1 \text{ when } x'^2 < x^2 \\ 0 \text{ when } x'^2 > x^2 \end{cases}.$$

[4] In order for us to be able to make use of the tables of integrals [7], it is convenient to differentiate the integral

$$\Phi(x) = \int_{r_0}^{\alpha - r_0} \ln \frac{r_0^2}{(x'-x)^2} e^{-\mu x'^2} \mu x'^2 dx'$$

with respect to the parameter x.