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THE GENERAL ORBIT
OF
HECTOR

BY

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INTRODUCTION

IN this investigation the general orbit of Hector is determined by the application of Professor E. W. Brown's theory*, giving expressions for the co-ordinates explicitly in terms of the time. The large amplitude of Hector's libration and the large inclination of its orbit make this asteroid particularly interesting. The first approximation to the values of the constants of the theory are obtained by the application of harmonic analysis to several osculating orbits. The second approximation is made with the help of approximate expressions obtained from the general orbit of Achilles. The notation used is, unless the contrary is explicitly stated, that of Professor Brown's memoir.

SECTION I. DETERMINATION OF THE CONSTANTS

1. The arbitrary constants of the solution to be determined are:

τ_1 = Amplitude of the libration.
ν_0 = Phase of the libration at epoch.
η = Constant of eccentricity somewhat analogous to that of the ordinary theory.
ϖ = Constant of Perihelion.
$\gamma = 1 - \cos i$, where i is the constant part of the inclination of the asteroid's orbit to that of Jupiter.
θ_0 = Constant part of the longitude of the node of these same planes.

2. The first approximation to the values of these constants was obtained from the eight sets of osculating elements of Strömgren and Vinter Hansen†. These elements being given for equidistant epochs during a period of seven years lent themselves readily to harmonic analysis and gave the constants with considerable accuracy.

The inclination and the longitude of the node were the easiest constants to determine, for they are comparatively independent of the four other constants, and, with the exception of being referred to the plane of Jupiter's orbit in one case and to the ecliptic in the other, play the same rôle in Brown's theory as in the osculating ellipse. The eight values of each of these quantities were plotted and the means taken from the graphs. These mean-values were then referred to the plane of Jupiter's orbit to give i and θ_0.

The constants of the libration and those of eccentricity and perihelion were troublesome because the expressions for one involve the values of the others, and successive approximations were necessary to determine each.

To obtain a first approximation to τ_1 and ν_0 I used the quantities obtained by subtracting from the mean longitude of Hector at each of Strömgren's epochs the corresponding mean

* "Theory of the Trojan Group of Asteroids," *Transactions of the Yale Observatory*, Vol. 3, Parts I, III; *Astronomical Journal*, Nos. 825, 826.

† *Publikationer og mindre Meddelelser fra Københavns Observatorium*, No. 12.

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longitude of Jupiter. In order to remove from these values terms with the period of Jupiter, they were analysed by the method of least-squares into the following form:

$$M_H - M_J = K_1 + K_2 t + K_3 t^2 + K_4 \sin\psi + K_5 \cos\psi.$$

Here t is the time, ψ the mean anomaly of Jupiter, and $K_1, \ldots K_5$ are the constants to be determined from the solution. By removing the terms with argument ψ, we have remainders very nearly equal to the values of τ at these dates. These quantities we have represented for this short interval of time by a power series, but for the theory we need the values of the coefficients of a Fourier series, which, during this interval, will give the same values of τ.

The eight approximate values of τ at date are to be expressed in the form given in § 83 of Brown's Theory:

$$\tau = 60° + \tau_0 + \tau_1 \cos\phi_0 + \tau_2 \cos 2\phi_0 + \ldots,$$

$$\phi_0 = \nu v + \nu_0,$$

where τ_1 and ν_0 are the arbitrary constants, and

$$\nu = \frac{3\sqrt{3}}{2}\sqrt{m}\,(1 + \tfrac{15}{2}m - \tfrac{3}{16}\tau_1^2 - \tfrac{2}{3}\gamma + \tfrac{43}{48}\eta^2 + \tfrac{7}{4}\beta + \tfrac{31}{24}\sqrt{3}\,\alpha),$$

$$\tau_0 = \tfrac{3}{8}\sqrt{3}\tau_1^2 - \tfrac{1}{6}\sqrt{3}\gamma + \tfrac{29}{24}\sqrt{3}\eta^2 + \tfrac{5}{2}\alpha,$$

$$\tau_2 = -\tfrac{1}{8}\sqrt{3}\tau_1^2, \qquad \tau_3 = \tfrac{5}{64}\tau_1^3,$$

$$\alpha = \eta e' \sin(\varpi - \varpi' - 60°),$$

$$\beta = e'^2 + \eta e' \cos(\varpi - \varpi' - 60°),$$

e', ϖ' = eccentricity and longitude of the perihelion of Jupiter's orbit.

Obviously such a representation must be made by successive approximations.

η and ϖ were determined from the following formulae:

$$e \begin{matrix}\cos\\ \sin\end{matrix}\varpi'' = \eta \begin{matrix}\cos\\ \sin\end{matrix}\varpi + e' \begin{matrix}\cos\\ \sin\end{matrix}(\tau_0 + 60° + \varpi'),$$

where e, ϖ'' are the mean values of the osculating eccentricity and the perihelion, and τ_0 is the value determined by the method of the preceding paragraph. Since the expressions for τ involve η, ϖ, while the expressions for η, ϖ involve τ_1, additional applications of the method of successive approximations were required.

The values of the constants obtained by this process were:

$$\gamma = \cdot 05335, \qquad \eta = \cdot 0537, \qquad \tau_1 = \cdot 328,$$

$$\theta_0 = 338° \, 32', \qquad \varpi = 219° \, 21', \qquad \nu_0 = 135° \, 10'.$$

3. The direct perturbations of Hector by Saturn were obtained from Leverrier's theory of Jupiter and Saturn* by the method outlined in Chapters III and IV of Brown's Theory. For this purpose were needed the mean values of the osculating elements of Hector. These were obtained with sufficient accuracy from the plotted values at Strömgren's epochs. The adopted mean values were

$$i = 18° \, 9' \, 30'', \qquad e = \cdot 0321, \qquad \varpi = 156° \, 32', \qquad \Omega = 341° \, 58'.$$

After the final values for the constants of the libration had been obtained the perturbations involving these constants were corrected.

* *Annales de l'Observatoire de Paris, Mémoires*, Vols. X, XI.

In view of the large amplitude of libration of Hector it was necessary to include its effect in the $2M'' - M$ terms in e_s and $e\varpi_s$. The effect of this was to slightly change the terms with argument $2M'' - M$ and to introduce terms with this argument plus and minus ϕ_0.

The chief difficulty with using the theory of Jupiter came from the large value of Hector's inclination, but the errors from this cause are within the limits of the accuracy obtained.

Some of these perturbations were applied in the same manner as had been done for Achilles, but it seemed advisable in the case of the eccentricity and perihelion to add the terms of long period to η, ϖ and those of short period to e, ϖ''. Also, those of the inclination and the node were added to the ecliptic values of these quantities rather than to i and θ of the theory.

The indirect perturbations were obtained by the method suggested by Brown and gave no trouble.

4. The perturbations by Saturn having been determined, it was possible to remove their effects from the elements; leaving the effects of Jupiter only. The elements thus obtained were then available for making the second approximation to the constants of the Trojan theory. In this approximation γ and θ_0 were obtained by the use of elliptical expressions; while τ_1, ν, η, and ϖ were determined by using the orbit of Achilles to give approximate expressions for that of Hector, as suggested in § 137 of Brown's Memoir.

In the Trojan theory the principal part of the inclination is given by

$$\Gamma = 1 - \cos i = \gamma q^{\frac{1}{2}},$$

where γ is the arbitrary constant. We can, to a considerable degree of approximation, replace q by the corresponding elliptic expression

$$q = \frac{a'}{a\,(1 - e^2)},\ \text{where } a' = \text{Jupiter's semi-major axis,}$$
$$a,\ e = \text{Hector's semi-major axis and eccentricity.}$$

Using for i the values obtained by referring the osculating inclination and longitude of the node to the plane of Jupiter's orbit, and using the values of a and e from the osculating orbits, eight values of γ were derived. These were plotted and the mean adopted. This value of γ is so near the true value that final changes did not affect any of the expressions involving this quantity. The value of θ_0 was obtained from the eight plotted values of the quantity, and gave no trouble.

Brown's Theory gives the orbital longitude by the expression

$$v = M + (\text{equation of centre with } M - \varpi'',\ e) + (\delta v - \delta t),$$

whereas in an osculating ellipse

$$v_{\text{osc}} = M_{\text{osc}} + (\text{equation of centre with } M_{\text{osc}} - \varpi_{\text{osc}},\ e_{\text{osc}}).$$

Here v and v_{osc} differ only by a small constant which arises from the fact that v is measured along the ecliptic, then along Jupiter's orbit, and finally along that of Hector, whereas v_{osc} is measured in two planes only. As δt and δv are small, M, n, e, ϖ'' do not differ much from the corresponding elliptic values. There are similar relations in the case of u, the reciprocal of the radius vector. The method used here was to determine the corrections δM, δn, δe, $\delta\varpi''$, which must be applied to the osculating elements to give the corresponding ones of the Trojan theory. These were obtained, as suggested by Brown, by solving the four linear equations

$$\left(\delta M \frac{\partial}{\partial M} + \delta e \frac{\partial}{\partial e} + \delta\varpi'' \frac{\partial}{\partial \varpi''} + \delta n \frac{\partial}{\partial n}\right) v,\ u,\ \dot{v},\ \dot{u} = \delta v,\ \delta u,\ \delta\dot{v},\ \delta\dot{u},$$

where the partial derivatives were obtained from elliptic expansions, and the right-hand members from the corresponding values of these quantities for Achilles. These latter quantities were computed by multiplying, in the expressions for Achilles, terms with argument $i\phi_0$ by $[\tau_1 \text{ (Hector)}/\tau_1 \text{ (Achilles)}]^i$ and those with argument $i\,(M - \varpi_1)$ by $[e_1 \text{ (Hector)}/e_1 \text{ (Achilles)}]^i$. These expressions were derived for each of Strömgren's epochs, and also for the set of elements for the epoch 1924 obtained from these by V. Hase* with special perturbations. These latter elements were not as accurate as the former, but in view of their later date had considerable weight in the determination of the constants of the libration. The addition of the values of δM, δn, δe, $\delta\varpi''$ to the osculating values for each date gave nine sets of values for M, n, e, ϖ'' from which to determine the four corresponding constants of the Trojan theory.

For the determination of the constants of the libration, the expressions with the time as the independent variable† were used.

$$\tau_t = \tau_0 + \tau_1 \cos\phi_0 + \tau_2 \cos 2\phi_0 + \ldots - \tfrac{2}{9}\tau_1{}^2 \nu \sin 2\phi_0,$$
$$\phi_0 = \overline{\phi}_0 + (299''{\cdot}128t + M_s - \overline{M}_s + H_8 - \overline{H}_8)\,\nu + \phi_s - \overline{\phi}_s,$$
$$M = +\,299''{\cdot}128t + \tau_t + M_s + H_8.$$

Here M_s, ϕ_s are direct, and H_8, indirect perturbations by Saturn, and t is the number of days from the epoch. A bar placed over a symbol indicates the value of that quantity at the epoch. The arbitrary constant ν_0 has been replaced by $\overline{\phi}_0$, whereas τ_1 has the same value as before.

The use of the orbit of Achilles for the determination of η, ϖ is evident from examination of the expressions for e, ϖ'' on pp. 125 and 133 of Brown's Memoir.

This method of using approximate developments of another asteroid does not seem to have been justified in this case, for the calculations were laborious and the gain in accuracy did not warrant the additional labour. Part of the difficulty was undoubtedly due to the very large value of τ_1 for Hector.

The results from the second approximation were:

$$\nu = 0{\cdot}053513, \quad \tau_1 = 0{\cdot}32863, \quad \eta = 0{\cdot}0476, \quad \overline{\phi}_0 = 154^\circ\, 29'\, 40'', \quad \varpi = 224^\circ\, 20',$$

τ_1 was not believed to be known with this accuracy but the constant b used in the theory, which is a function of τ_1, was adopted as 0·26300 and this corresponded to the above value of τ_1. It is desirable to carry extra decimal places so that the differential corrections to be applied later will be significant.

Section II. CALCULATION OF THE EXPRESSIONS FOR THE CO-ORDINATES

5. The intermediate orbit was computed by harmonic analysis using the general formulae of § 142 and the values of the constants from the preceding paragraph. The following corrected expressions were substituted for those in the text:

$$\Delta^2 = 2 - (2 - \Gamma)\cos\tau = d^2\,(1 + b\cos\phi)^2,$$
$$C_3 = \frac{3}{2\Delta^5}\,s^2 - \left(1 + \frac{1}{2\Delta^3}\right)c,$$
$$E = \frac{105}{\Delta^9}\,s_4 - \frac{90}{\Delta^7}\,s^2c - \frac{12}{\Delta^5}\,s^2 + \frac{9}{\Delta^5}\,c^2 - \left(1 - \frac{1}{\Delta^3}\right)c.$$

* *B.I.A.*, Leningrad, No. 8.

† § 149 of Brown's Memoir.

Because of the large value of b for Hector, the functions were computed for nine values of ϕ instead of seven as suggested in § 21; the values for $\phi = 45°$ and $135°$ being added.

As a check on the calculations equation (142·9) was solved for the constant $\mathscr{C}$, and this quantity computed for the special values of ϕ. These values were found to agree among themselves and with the values of $\mathscr{C}$ computed by equations (71·1) and (71·2) with the expected accuracy.

The calculation of R_b and R'', involving x, was necessarily performed by successive approximations. The following arrangement of equations (142·3), (142·5), and (142·8) was used:

$$m\left(\mathscr{C} - \frac{6R_0}{m} + \chi\right)(1 - \tfrac{5}{9}x)^2 = x^2\,(1 + Q + \tfrac{5}{27}x^2),$$

where $Q = R'' - \frac{5}{27}x^2$ with the expression for R'' taken from (142·3). It will be noticed that this equation does not contain x implicitly, and all the other quantities involved can be computed for the nine values of ϕ. The value of $\mathscr{C}$ was taken as the mean of the values from (71·1) and (71·2). x was then determined by approximation to satisfy this equation.

The nine special values of the integrand (142·10) were analysed by an adaptation of the method of § 21, and the series integrated term by term. This series was inverted by the scheme of § 23 to express ϕ as a function of ϕ_τ. As a cosine series in ϕ_τ, τ was obtained by harmonic analysis of special values determined from the relation

$$2 - (2 - \Gamma)\cos\tau = d^2\,(1 + b\cos\phi)^2.$$

Then, as

$$\phi_\tau = \phi_0 - \tfrac{5}{9}\nu\,(\tau - \tau_0),$$

τ could be expressed, by successive approximations, as a cosine-sine series in ϕ_0. As a check the values of τ were computed for $\phi_0 = 15°, 30°, 45°, \ldots 180°, 270°$; which values checked within 2″. As a final check were computed the two relations of § 85:

$$d^2\,(1 + b^2) = 1 + \tfrac{1}{2}\gamma + \tfrac{1}{2}\tau_1^2 + (\tau_0 + \tau_2)\sqrt{3},$$

$$2d^2b = \tau_1\,(\sqrt{3} - \tfrac{1}{2}\sqrt{3}\gamma + \tau_0 + \tau_2) + (\tau_3 - \tfrac{1}{6}\tau_1^3)\sqrt{3},$$

these agreed to 5 and 4 significant figures, which is all that can be expected when we assume as we did in this test that the mean value of $\Gamma = \Gamma_{\phi=0} = \Gamma_{\phi=\pi}$.

6. The calculation of E was carried out as suggested in Section III of Brown's Theory. From the expression for τ as a function of ϕ_0 obtained from the intermediate orbit were computed the special values of τ, x. The functions were computed for $\phi_0 = 0°, 180°, 90°, 270°, 60°, 300°, 120°, 240°$ to fit a convenient scheme of analysis and synthesis. The only new operation involved here was the multiplication of series, which was carried out by harmonic analysis, and by direct multiplication of the coefficients as a check.

Before computing the third-order terms it was desirable to have a more accurate value of η. This was obtained by computing e, ϖ'' with the expressions thus far obtained, and comparing them with Strömgren's values. The improved values of η, ϖ were used in the subsequent work.

The third-order terms are given by § 146,

$$E_3 = -\tfrac{1}{2}D_0^{-1}\,D_1P'',$$

where

$$P'' = P_0''\cos(v - \varpi_1) + P_1''\sin(v - \varpi_1),$$

P_0'', P_1'' being Fourier series in ϕ_0, and where D_1 operates on terms with the period of Jupiter and D_0^{-1} on terms of long period. This can therefore be expressed thus:

$$E_3 = -\tfrac{1}{2}D_0^{-1}(A)\sin v - \tfrac{1}{2}D_0^{-1}(B)\cos v,$$
$$A = -P_0''\cos\varpi_1 + P_1''\sin\varpi_1,$$
$$B = P_0''\sin\varpi_1 + P_1''\cos\varpi_1.$$

Now E_3 is a small addition to E,

$$E_3 = \delta E = \delta\,[e\cos(v - \varpi'')]$$
$$= \delta\,(e\sin\varpi'')\sin v + \delta\,(e\cos\varpi'')\cos v,$$

whence

$$\delta\,(e\sin\varpi'') = -\tfrac{1}{2}D_0^{-1}(A),$$
$$\delta\,(e\cos\varpi'') = -\tfrac{1}{2}D_0^{-1}(B).$$

These terms can be added to η, ϖ by means of the relations

$$\delta\,(\eta\cos\varpi) = \delta\,(e\cos\varpi''),$$
$$\delta\,(\eta\sin\varpi) = \delta\,(e\sin\varpi'').$$

The expressions A and B were computed, analysed, and integrated, and the resulting additions to η, ϖ were added to e_0, ϖ.

7. The expressions for δt, δv, p, q, γ, θ gave few new difficulties. The formulae of § 147 were followed rather closely and the principal task was to avoid the calculation of insensible terms, and to express the final results in convenient form.

8. The final expressions for the co-ordinates are tabulated separately in Section IV for the convenience of those who may wish to use them for purposes of prediction. The use of these formulae for practical application is obvious, but we will explain here the origin of some of the terms involved.

The notation employed is that of Brown's Memoir. Quantities with the subscript s are due to the direct action of Saturn. H_8 and the last term in e_1' and e_2' contain the indirect action of this planet.

The terms in e_s, $e\varpi_s$ with argument $2M'' - M$ and with this argument plus and minus ϕ_0 include the effect of the libration on these terms.

The true longitude, as used in the theory and denoted by v, and the radius vector r are first computed. After computing i, θ of the theory they are referred to the ecliptic and the perturbations by Saturn added. With these and the value of v obtained above the ecliptic longitude and latitude are computed.

Section III. COMPARISON WITH OBSERVATIONS AND DIFFERENTIAL CORRECTION OF THE CONSTANTS

9. The comparison was first made with heliocentric positions computed from the osculating orbits of Strömgren and Hase, and the more troublesome comparison with geocentric positions left until later.

For each epoch of Strömgren's eight sets of elements and for the Leningrad epoch the longitude in the orbit was computed. These longitudes from Strömgren's elements are probably correct to less than 10″ and extend over half the period of Jupiter. They should therefore give the eccentricity and perihelion with considerable accuracy. The 1924 set of elements is not so accurate, but at that epoch the error will effect principally the libration and will be easily corrected in the later comparison. It was therefore decided so to adjust the constants as to fit these 9 values of the longitude as accurately as possible. The theoretical values were computed and residuals as large as a degree were found. Approximate equations of the form

$$\delta v = \frac{\partial \nu}{\partial \tau_1}\,\delta\tau_1 + \frac{\partial \nu}{\partial \bar{\phi}_0}\,\delta\bar{\phi}_0 + \frac{\partial \nu}{\partial \eta}\,\delta\eta + \frac{\partial \nu}{\partial \varpi_0}\,\delta\varpi_0$$

were calculated for each of the nine epochs. Theoretically these should be solved by the method of least squares, but in practice it was found expedient to solve four of them and then by inspection change the solution to give as good a fit as possible. These equations were solved with the residuals mentioned above as δv's, and the resulting $\delta\tau_1$, $\delta\bar{\phi}_0$, $\delta\eta$, $\delta\varpi_0$ were found to satisfy the equations within about 30″. These corrections to the four arbitrary constants were then applied differentially to the expansions, and the value of v again computed, this time with residuals of the order of 10′. The fourth repetition of this process gave no residual greater than 15″.

The inclinations were next computed and compared with those of Strömgren, giving residuals which were systematically too large by about 10″. This part was easily removed by a very slight change in γ which was much too small to have any effect on the developments involving γ. After this change the maximum residual was 6″.

A similar comparison of θ showed this quantity to have a systematic error of 1′. This was easily removed, leaving residuals up to 20″.

10. The final comparison was made with seven geocentric positions: the three normal places II, V, and XII given on page 9 of Strömgren's Memoir, two observations from Vienna, and two from Yerkes. This comparison gave residuals less than 5′. After the constants were differentially corrected there remained the residuals in the following table:

Date (U.T.)	$\Delta\alpha \cos \delta$	$\Delta\delta$	Observation	Reference
1907 March 7·8861	+ 1″·6	− 1″·5	Strömgren II	Footnote, p. 159
1908 March 2·4924	+ 1·1	− ·9	,, V	,,
1912 Aug. 11·8186	− ·2	− ·2	,, XII	,,
1914 Sept. 26·8510	− 1·5	− 1·1	Vienna	A.N. 4781
1919 Feb. 25·0068	− 1·7	+ 1·6	,,	A.N. 4989
1926 Oct. 8·1376	+ 1·0	+ ·8	Yerkes	A.J. 891
1927 Oct. 20·2210	+ ·9	+ ·4	,,	A.J. 912

These residuals are slightly larger than was hoped for and seem to be systematic; but, considering the very large value of the amplitude of libration, they are quite satisfactory. They indicate the presence of a term with a period approximately half that of the libration, but the interval of time covered by the observations is not yet long enough to give definite information. The differential corrections which were made after the general calculation was finished were fairly large and may have contributed to the residuals. However, as any great gain in accuracy in the determination of the constants of libration can be obtained only after more time has elapsed, a general recalculation of the orbit would not now be profitable.

SECTION IV. FORMULAE GIVING THE HELIOCENTRIC CO-ORDINATES OF HECTOR EXPLICITLY IN TERMS OF THE TIME

11. The values of the co-ordinates of Hector for any date may be obtained by substituting in the following expressions the value of t at that date. These equations give the heliocentric radius vector r, the longitude on the ecliptic v_{ecl}, and the latitude above the ecliptic b_{ecl}, referred to the equinox of 1910·0. The terms are arranged in the order in which they will be needed for calculation.

t = the number of mean solar days from the epoch May 12·4628 G.C.T., 1910.

$$M_s = \Sigma C \sin \text{Arg.}$$

Arg. Source	Arg. Value	C
$M'' - M$	$25° + 0°.0496t$	32″
$2M'' - 2M$	$230 + \cdot 0993t$	57
$3M'' - 3M$	$74 + \cdot 1489t$	19
$4M'' - 4M$	$280 + \cdot 1985t$	7
$5M'' - 5M$	$124 + \cdot 2481t$	3
M''	$153 + \cdot 0335t$	4
$2M'' - M$	$8 + \cdot 0162t$	83
$3M'' - 2M$	$268 + \cdot 0658t$	19
$4M'' - 3M$	$96 + \cdot 1154t$	7
$5M'' - 4M$	$306 + \cdot 1651t$	4
$2M''$	$294 + \cdot 0669t$	8
$2M$	$343 + \cdot 1662t$	5
$M'' + M$	$138 + \cdot 1166t$	5
$3M'' - M$	$235 + \cdot 0173t$	15
$4M'' - 2M$	$101 + \cdot 0323t$	13
$5M'' - 2M$	$176 + \cdot 0011t$	98
$10M'' - 4M$	$112 + \cdot 0021t$	5
$2M'' - M + \phi_0$	$347 + \cdot 0100t$	89
$2M'' - M - \phi_0$	$111 + \cdot 0226t$	11
$3M'' - M + \phi_0$	$76 + \cdot 0236t$	2
$3M'' - M - \phi_0$	$312 + \cdot 0110t$	14
$5M'' - 2M + \phi_0$	$214 + \cdot 0074t$	9
$5M'' - 2M - \phi_0$	$270 + \cdot 0052t$	10

$$\phi_s = \Sigma C \sin \text{Arg.}$$

Arg. Source	Arg. Value	C
$2M'' - M$	$299° + 0°.0162t$	250″
$3M'' - M$	$124 + \cdot 0173t$	50
$5M'' - 2M$	$227\cdot 6 + \cdot 00106t$	1050
$10M'' - 4M$	$145 + \cdot 0021t$	30

$$\tau_s = \Sigma C \sin \text{Arg.}$$

Arg. Source	Arg. Value	C
$2M'' - M$	$294° + 0°.0162t$	23″
$2M'' - M + \phi_0$	$322 + \cdot 0099t$	49

H_8 = (tabular value VIII − 12·6285) 102″.571, where "tabular value VIII" is that of table VIII of Hill's Tables of Jupiter*,

$$\phi_0 = 156^\circ\ 00'\ 15'' + (299''.1284t + M_s + H_8)\ {\cdot}076096 + \phi_s,$$

$$\tau_t = 63^\circ\ 36'\ 23'' + 69256'' \cos\phi_0 - 4800'' \cos 2\phi_0 + 540'' \cos 3\phi_0 - 78'' \cos 4\phi_0 + 13'' \cos 5\phi_0 + 50'' \sin\phi_0 - 607'' \sin 2\phi_0 + 129'' \sin 3\phi_0 - 21'' \sin 4\phi_0 + \tau_s - ''.0041t,$$

$$M = 192^\circ\ 36'\ 13'' + 299''.1284t + \tau_t + M_s + H_8.$$

$$\eta_s = \Sigma C \cos \text{Arg.}$$

Arg. Source	Arg. Value	C
$5M'' - 2M - 3\varpi$	$292^\circ + 0^\circ.00026t$	60″
$5M'' - 2M - 2\varpi$	$227 + {\cdot}00052t$	68
$5M'' - 2M - \varpi$	$12 + {\cdot}00079t$	112

$$\varpi_s = 19\Sigma\ C \sin \text{Arg.}$$

Arg. Source	Arg. Value	C
$5M'' - 2M - 3\varpi$	$292^\circ + 0^\circ.00026t$	60″
$5M'' - 2M - 2\varpi$	$227 + {\cdot}00052t$	68
$5M'' - 2M - \varpi$	$12 + {\cdot}00079t$	112

$$e_s = \Sigma\ C \cos \text{Arg.}$$

Arg. Source	Arg. Value	C
M''	$63^\circ + 0^\circ.0335t$	4″
$2M'' - M$	$344 + {\cdot}0162t$	105
$2M'' - M + \phi_0$	$278 + {\cdot}0098t$	29
$2M'' - M - \phi_0$	$230 + {\cdot}0224t$	13
$3M'' - 2M$	$173 + {\cdot}0658t$	18
$4M'' - 3M$	$18 + {\cdot}1154t$	6
$5M'' - 4M$	$223 + {\cdot}1651t$	3
$3M'' - M$	$177 + {\cdot}0173t$	18
$4M'' - 2M$	$208 + {\cdot}0323t$	10
$5M'' - 3M$	$.65 + {\cdot}0820t$	4
M	$268 + {\cdot}0831t$	2
$-2M'' + 3M$	$131 + {\cdot}3151t$	3

$$e\varpi_s'' = \Sigma\ C \sin \text{Arg.}$$

Arg. Source	Arg. Value	C
M''	$56^\circ + 0^\circ.0335t$	15″
$2M'' - M$	$164 + {\cdot}0162t$	121
$2M'' - M + \phi_0$	$98 + {\cdot}0098t$	33
$2M'' - M - \phi_0$	$50 + {\cdot}0224t$	15
$3M'' - 2M$	$353 + {\cdot}0658t$	18
$4M'' - 3M$	$198 + {\cdot}1154t$	6
$5M'' - 4M$	$43 + {\cdot}1651t$	3
$3M'' - M$	$177 + {\cdot}0173t$	18
$4M'' - 2M$	$28 + {\cdot}0323t$	10
$5M'' - 3M$	$245 + {\cdot}0820t$	4
M	$262 + {\cdot}0831t$	8
$-2M'' + 3M$	$131 + {\cdot}3151t$	3

* *Astronomical Papers of the American Ephemeris*, Vol. VII.

$$
\begin{aligned}
e_0 = {} & \cdot 05325 + \cdot 00010 \cos \phi_0 - \cdot 00002 \cos 2\phi_0 + \cdot 00001 \cos 3\phi_0 \\
& + \cdot 00006 \sin \phi_0 - \cdot 00004 \sin 2\phi_0 + \cdot 00002 \sin 3\phi_0 \\
& + \eta_s/206265 - \cdot 0000155 \, \frac{M - 238 \cdot 0}{57 \cdot 3},
\end{aligned}
$$

$$
\begin{aligned}
\varpi = {} & 214^\circ \, 04'\!\cdot 5 - 7'\!\cdot 0 \cos \phi_0 + 2'\!\cdot 4 \cos 2\phi_0 - '\!\cdot 4 \cos 3\phi_0 + '\!\cdot 2 \cos 4\phi_0 \\
& - 94'\!\cdot 5 \sin \phi_0 + 12'\!\cdot 8 \sin 2\phi_0 - 2'\!\cdot 3 \sin 3\phi_0 + '\!\cdot 6 \sin 4\phi_0 \\
& + \cdot 002732 \, (M - 238 \cdot 0) + \varpi_s + ''\!\cdot 016t,
\end{aligned}
$$

$$
\varpi' = 12^\circ \, 52'\!\cdot 6 + ''\!\cdot 0211t,
$$

$$
\begin{aligned}
e_1' = {} & + \cdot 04525 - \cdot 00002 \cos \phi_0 + \cdot 00001 \cos 3\phi_0 + \cdot 00013 \cos ({}^\circ\!\cdot 00105t + 248^\circ) \\
& + \cdot 00070 \sin \phi_0 - \cdot 00015 \sin 2\phi_0 + \cdot 00003 \sin 3\phi_0,
\end{aligned}
$$

$$
\begin{aligned}
e_2' = {} & - \cdot 00353 - \cdot 00006 \cos \phi_0 + \cdot 00006 \sin \phi_0 + \cdot 00003 \sin 2\phi_0 \\
& - \cdot 00001 \sin 3\phi_0 + \cdot 00013 \sin ({}^\circ\!\cdot 00105t + 248^\circ),
\end{aligned}
$$

$$
e \begin{matrix}\cos \\ \sin\end{matrix} \varpi'' = e_0 \begin{matrix}\cos \\ \sin\end{matrix} \varpi + e_1' \begin{matrix}\cos \\ \sin\end{matrix} (\varpi' + 60^\circ) + e_2' \begin{matrix}-\sin \\ \cos\end{matrix} (\varpi' + 60^\circ),
$$

$$
e_1 \begin{matrix}\cos \\ \sin\end{matrix} \varpi_1 = e \begin{matrix}\cos \\ \sin\end{matrix} \varpi'' - \cdot 0484 \begin{matrix}\cos \\ \sin\end{matrix} (\varpi' + \tau_t),
$$

$$
\begin{aligned}
\delta t_1 = {} & (-27'' \cos \phi_0 + 8'' \cos 2\phi_0) \cos (M - \varpi' - \tau_t) + (43'' - 8'' \cos \phi_0) \cos (M - \varpi_1) \\
& + (-223'' + 218'' \cos \phi_0 - 71'' \cos 2\phi_0 + 21'' \cos 3\phi_0 - 6'' \cos 4\phi_0) \sin (M - \varpi_1),
\end{aligned}
$$

$$
\begin{aligned}
\delta t_2 = {} & (+5'' - 6'' \cos \phi_0) \sin (2M - \varpi' - \tau_t - \varpi_1) + (-10'' + 11'' \cos \phi_0 - 5'' \cos 2\phi_0) \\
& \cos (2M - \tau_t + 43^\circ) + (5'' \cos \phi_0) \sin (2M - \tau_t + 43^\circ),
\end{aligned}
$$

$$
\begin{aligned}
\delta v = {} & (+23'' - 25'' \cos \phi_0 + 10'' \cos 2\phi_0) \sin 2 \, (M - \varpi_1) \\
& + (19'' - 19'' \cos \phi_0 + 5'' \cos 2\phi_0) \sin (2M - \varpi' - \tau_t - \varpi_1),
\end{aligned}
$$

$v = M - (\delta t_1 + \delta t_2) + \delta v +$ equation of centre with eccentricity $(e + e_s)$ and mean anomaly $[M - (\varpi'' + \varpi_s'')]$.

$$
q_s = 10^{-5} \, \Sigma \, C \cos \text{Arg}.
$$

Arg.		
Source	Value	C
$M'' - M$	$155^\circ + 0^\circ\!\cdot 0496t$	3
$3M'' - 3M$	$74 + \cdot 1489t$	5
$4M'' - 4M$	$279 + \cdot 1985t$	2
$2M'' - M$	$230 + \cdot 0993t$	12
$5M'' - 2M$	$24 + \cdot 0011t$	14
$2M'' - M + \phi_0$	$187 + \cdot 0100t$	4
$2M'' - M - \phi_0$	$349 + \cdot 0226t$	4
$5M'' - 2M + \phi_0$	$147 + \cdot 0074t$	5
$5M'' - 2M - \phi_0$	$296 + \cdot 0052t$	3

$$
\begin{aligned}
\tfrac{2}{3}x = {} & - \cdot 00024 + \cdot 00008 \cos \phi_0 - \cdot 00008 \cos 2\phi_0 + \cdot 00003 \cos 3\phi_0 - \cdot 00002 \cos 4\phi_0 \\
& - \cdot 01704 \sin \phi_0 + \cdot 00236 \sin 2\phi_0 - \cdot 00040 \sin 3\phi_0 + \cdot 00008 \sin 4\phi_0 - \cdot 00002 \sin 5\phi_0 \\
& + (- \cdot 0013 \cos \phi_0 + \cdot 0004 \cos 2\phi_0 - \cdot 0001 \cos 3\phi_0) \\
& \qquad \times [2e \sin (M - \varpi'') + \tfrac{5}{4}e^2 \sin (2M - 2\varpi'')],
\end{aligned}
$$

$$pq = \frac{1 + \frac{2}{3}x}{1 - e^2} + (\cdot 00022 + \cdot 00006 \cos \phi_0 - \cdot 00016 \cos 2\phi_0 + \cdot 00006 \cos 3\phi_0$$
$$+ \cdot 00003 \sin 2\phi_0 - \cdot 00003 \sin 3\phi_0 + \cdot 00003 \sin 4\phi_0)$$
$$+ (\cdot 00012 + \cdot 00010 \cos \phi_0 - \cdot 00004 \cos 2\phi_0) \sin (M - \varpi_1)$$
$$+ (\cdot 00046 - \cdot 00043 \cos \phi_0 + \cdot 00014 \cos 2\phi_0 - \cdot 00004 \cos 3\phi_0) \cos (M - \varpi_1)$$
$$+ (\cdot 00006 \cos \phi_0) \sin (M - \tau_t - \varpi')$$
$$+ (\cdot 00003 - \cdot 00004 \cos \phi_0) \cos 2 (M - \varpi_1)$$
$$+ (- \cdot 00003 + \cdot 00003 \cos \phi_0) \sin (2M - \tau_t + 43^\circ)$$
$$+ (- \cdot 00050 + \cdot 00040 \cos \phi_0) \sin (M - \varpi_1) [2e \sin (M - \varpi'')]$$
$$+ q_s,$$

$$\frac{1}{r} = u = \frac{pq}{5 \cdot 2028} [1 + e \cos (v - \varpi'')],$$

$$\sqrt{2} \sin i/2 = \cdot 23125 (pq)^{\frac{1}{2}} + \cdot 00002 \cos \phi_0 - \cdot 00004 \sin \phi_0$$
$$- \cdot 00004 \cos \phi_0 \cos (2M - \tau_t + 43^\circ)$$
$$+ (\cdot 00002 - \cdot 00002 \cos \phi_0) \sin (M - \tau_t + 43^\circ + \varpi_1)$$
$$+ \cdot 000002 \frac{(M - 238 \cdot 0)}{57 \cdot 3},$$

$$\theta = 338^\circ\, 28' \cdot 5 - \cdot 000093 (M - 238 \cdot 5)$$
$$+ 640'' \sin \phi_0 - 180'' \sin 2\phi_0 + 40'' \sin 3\phi_0 - 10'' \sin 4\phi_0$$
$$+ (- 20'' + 40'' \cos \phi_0 - 20'' \cos 2\phi_0) \cos (M - \varpi_1)$$
$$+ (- 40'' \cos \phi_0 + 20'' \cos 2\phi_0) \sin (2M - \tau_t + 43^\circ)$$
$$+ (- 20'' + 20'' \cos \phi_0) \cos (M - \tau_t + 43^\circ + \varpi_1),$$

$$\frac{i'}{2} = 39'\, 15'',$$

$$\theta' = 99^\circ\, 33'\, 15'' + {}''\!\!.018t,$$

$$\tan \tfrac{1}{2} (\alpha_1 + \alpha_2) = \frac{\cos \frac{1}{2} (i - i')}{\cos \frac{1}{2} (i + i')} \tan \tfrac{1}{2} (\theta - \theta'),$$

$$\tan \tfrac{1}{2} (\alpha_1 - \alpha_2) = \frac{\sin \frac{1}{2} (i - i')}{\sin \frac{1}{2} (i + i')} \tan \tfrac{1}{2} (\theta - \theta'),$$

$$\tan \tfrac{1}{2} (i_{\text{ecl}} - i_s) = \frac{\cos \frac{1}{2} (\alpha_1 + \alpha_2)}{\cos \frac{1}{2} (\alpha_1 - \alpha_2)} \tan \tfrac{1}{2} (i + i').$$

$$i_s = \Sigma\, C \cos \text{Arg}.$$

Arg.		
Source	Value	C
$2M''$	$114^\circ + 0^\circ\!.0669t$	$6''$
$3M'' - M$	$269 + \cdot 0173t$	15
$4M'' - 2M$	$116 + \cdot 0323t$	5
$5M'' - 2M$	$22 + \cdot 0011t$	53

$$\theta_s = -\,''\!.0157t + 10\Sigma\, C \sin \text{Arg.}$$

Arg. Source	Arg. Value	C
$M'' - M$	$25^\circ + 0^\circ\!.0496t$	$3''$
$M'' - M$	$230 + \cdot 0993t$	2
$2M'' - 2M$	$13 + \cdot 0831t$	1
$2M'' - M$	$56 + \cdot 0162t$	2
$2M''$	$114 + \cdot 0669t$	2
$3M'' - M$	$269 + \cdot 0173t$	4
$4M'' - 2M$	$284 + \cdot 0323t$	2
$5M'' - 2M$	$304 + \cdot 0011t$	13

$$\theta_{\text{ecl}} = \theta' + \alpha_1 + \theta_s,$$

$$\tan(v_{\text{ecl}} - \theta_{\text{ecl}}) = \tan(v - \theta + \alpha_2)\cos i_{\text{ecl}},$$

$$\sin b_{\text{ecl}} = \sin(v - \theta + \alpha_2)\sin i_{\text{ecl}}.$$