Extracting oscillation frequencies from sparse spectra: Fourier analysis

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Abstract

I begin by explaining the properties of spectral windows of time-series data. Emphasis is on data obtained at a single geographic longitude, but ground-based multi-longitude campaigns and space missions such as MOST and Hipparcos are not entirely neglected. In the second section, I consider the Fourier transform of time-series data and the procedure of pre-whitening. Sect. 3 is devoted to the pioneers of the subject. In Sect. 4, I suggest how to avoid pitfalls in the practice of periodogram-analysing variable-stars observations. In the last section, I venture an opinion.

Individual Objects: AR Her, δ Cet, β CMa, δ Sct, DD Lac, EN Lac, 2 And

Spectral windows

Spectral windows, also called window functions, are Fourier transforms of the observing windows. For a single site, the proof is illustrated in Fig. 1. This figure is a slight modification of figure 3-7 from the well-known monograph by Gray (1976). In the monograph, the figure serves to explain the working of a diffraction grating. In Fig. 1, the observing window is shown at top left, and its Fourier transform, at top right. The observing window consists of nine successive nights, equally spaced, each of duration Δt. It can be looked at as a result of a convolution of a rectangular function of width Δt (representing a single night) with a Shah function with spacing equal to the sidereal day, T_s, multiplied by another rectangular function of width T, equal to the total time-span of the observations. This wide rectangular function transforms into the narrow sinc function of width 1/T (bottom right), while the single-night rectangular function transforms into the wide sinc function of width 1/Δt (upper right). The latter, multiplied by the Shah function with spacing equal to 1/T_s (the transform of the Shah function at left) and convolved with the former gives the spectral window. Note that (1) the frequency resolution is determined by T, the total time-span of the data, and (2) 1/T_s is equal to one cycle per sidereal day (c/sd), i.e., 1.0027 c/d.

In Fig. 1, the diagram at lower left and all diagrams at right (i.e., in the frequency domain) are incomplete. In fact, the sinc and Shah functions extend from −∞ to +∞. In particular, the spectral window (top right) is a sum of an infinite series of the narrow sinc functions, spaced 1 c/sd, with their maximum ordinates modulated by the wide sinc function. The pattern has a maximum at zero frequency. Note that the duration of the observing night, Δt, determines the height of the −1.0027 and +1.0027 c/d side-lobes relative to the central peak: longer nights produce lower side-lobes.

The observing window in Fig. 1 (top left) is grossly simplified. I assumed that (1) on any night, the observations are taken continuously, (2) the nights are of equal duration, (3) there are no nights lost because of clouds or equipment failure. The first assumption was made to
avoid discussing the Nyquist frequency, an issue which for irregularly spaced time series seems to be debatable (see Koen 2006). Assumptions (2) and (3) make the observing window an even function of time, so that the spectral window is a real (and even) function of frequency. For actual data, the spectral window is a complex function. In practice, one plots the modulus of the spectral window. The modulus of a spectral window is an even function of frequency.
Figure 2: Synthetic amplitude spectrum of the MOST observations of \( \delta \) Cet. The arrows indicate aliases at \( f_{\text{orb}} - 6.2059 \) and \( f_{\text{orb}} + 6.2059 \) c/d, where \( f_{\text{orb}} = 14.20 \) c/d is the orbital frequency of the satellite.

An interesting example of how much a real spectral window may differ from the simplified case discussed above can be found in Borkowski’s (1980) analysis of visual observations of the RRab-type Blazhko variable AR Her, obtained in 1944 by Tsessevich. In the spectral window of these data (Borkowski’s figure 1) one can see not only the peak at zero frequency and the 1, 2, 3 etc. c/sd aliases, but also a peak at the star’s fundamental frequency of 2.128 c/d and its sidereal day aliases. This latter pattern arose because Tsessevich has spaced his observations unevenly: the sharp light-maxima he sampled with a much shorter time step than the flat minima.

The sidereal day aliases can be reduced or even eliminated altogether by observing from several sites at different longitude or from space. An example of a multi-longitude ground-based spectral window can be seen in the top panel of figure 2 of Handler et al. (2004), while an example of a space-based spectral window is shown in Fig. 2. Strictly speaking, both figures show moduli of the Fourier transforms of synthetic data, produced by sampling a sine-curve that represents the highest-amplitude variation of the star in question at the epochs of actual observations. In order to avoid confusion with the spectral windows proper, I shall refer to the former as “the synthetic spectra”. The synthetic spectrum in Fig. 2 was computed for the 6.2059 c/d variation of the \( \beta \) Cephei-type star \( \delta \) Cet; the synthetic data were produced by sampling a 13.8 mmag sine-curve of this frequency at the epochs of the MOST\(^1\) photometric observations of the star. The figure shows the 13.8 mmag peak at 6.2059 c/d and, in addition, much lower peaks at \( f_{\text{orb}} - 6.2059 \) and \( f_{\text{orb}} + 6.2059 \) c/d, where \( f_{\text{orb}} = 14.20 \) c/d is the orbital frequency of the satellite. For a detailed discussion of ground-based and MOST observations of \( \delta \) Cet, see Jerzykiewicz (2007).

In the Hipparcos’ (ESA 1997) epoch photometry, the satellite’s rotation-frequency aliases are more pronounced than the orbital-frequency aliases in the MOST observations of \( \delta \) Cet. In addition, the amplitude of the aliases is modulated with a frequency resulting from beating between the two sampling frequencies of the satellite. A thorough discussion of the spectral window of the Hipparcos’ epoch photometry has been provided by Jerzykiewicz & Pamyatnykh (2000).

\(^1\)The MOST satellite is a Canadian Space Agency mission, jointly operated by Dymacon Inc., the University of Toronto Institute for Aerospace Studies and the University of British Columbia, with the assistance of the University of Vienna.
Fourier transform of the observations and pre-whitening

Time-series observations of a variable star can be represented as a sum of (1) the product of the star’s intrinsic variation and the observing window, and (2) the observational noise. From the convolution theorem and the fact that the Fourier transform of a sine-curve of frequency $f$ is a pair of the $\delta$ functions, one placed at $-f$, and the other, at $+f$, it follows that:

T1 If the intrinsic variation is a sine-wave of frequency $f$ and amplitude $A$, the Fourier transform of the observations is equal to the sum of (1) the spectral window shifted to $-f$, (2) the spectral window shifted to $+f$, and (3) the Fourier transform of the observational noise; (1) and (2) are scaled to $A$.

T2 If the intrinsic variation is a sum of $N$ sine-waves, the Fourier transform of the observations is equal to the sum of $N$ of (1), $N$ of (2) and (3).

In practice, one plots the moduli of the Fourier transforms. T1 explains the orbital-frequency aliases in Fig. 2, in particular, the alias at $f_{\text{orb}} = 6.2059$ m/s. T2 makes it clear why pre-whitening is necessary if more than one frequency is present in the variation. If done in the time domain, as is usually the case, pre-whitening is an operation on real numbers. In the frequency domain, the real and the imaginary part of the transform must be treated. An example of pre-whitening in the time domain can be found in Handler et al. (2004). Pre-whitening in the frequency domain has been advocated by Gray & Desikachary (1973).

The pioneers

Meyer (1934) discovered that the radial velocity amplitude of the $\beta$ Cephei-type star $\beta$ CMa varied with a period equal to 49.1 d. He explained this in terms of an interference between two sine-curves of slightly different short periods, one equal to $6^h 0^m$, the other, to $6^h 2^m$. The reality of the components was supported by the fact that the longer of the two short periods was identical with the period of the variation of the width of spectral lines, discovered earlier by Henrotheau (1918).

Sterne (1938) applied the correct procedure of pre-whitening (without using the term) in order to derive a secondary period of $\delta$ Sct from photoelectric observations of Fath (1935, 1937).

Fath (1947) made an (unsuccessful) attempt to organize a multi-longitude campaign; the intended object was the $\beta$ Cephei-type star 12 (DD) Lac. His secondary period of 12 Lac, derived from a single-longitude data, was later shown to be a $\sim 1$ c/sd alias of the correct value. In 1956, de Jager (1963) organized the first successful multi-longitude campaign; the object was again 12 Lac.

Wehlau & Leung (1964) explained periodogram analysis in terms of Fourier transform and the convolution theorem.

Fellgett (1972) discussed limitations of periodogram analyses of time-series observations. He pointed out that (1) the existence of a Fourier component does not of itself provide any evidence of significant periodicity, (2) there is no unique Fourier representation of a function known over a finite interval of its argument. According to the NASA’s ADS, Fellgett’s (1972) important paper has been quoted only once. Apparently, Cassandras are unpopular.

Recommendations

This section should be skipped by those who do not make mistakes. The less fortunate among us may wish —before sending the results of their analysis to the editor— to go through the following list:
Figure 3: The radial-velocity observations (circles) and synthetic velocity-curve (solid line) of 16 (EN) Lac on JD 2451453. The observations and parameters of the synthetic curve are from Lehmann (2001). The synthetic curve was computed by MJ.

- Get the epochs of observations right. However, if you are lucky, your mistake may surface during the analysis. For an example, see figure 5 in Jerzykiewicz & Wenzel (1977).

- If you do differential photometry, use two (or more) comparison stars. A single comparison star may spoil the analysis. Example: I ascribed the frequency of 7.194 c/d to 16 (EN) Lac (Jerzykiewicz 1993). As it turned out, it was 2 And, the only comparison star I used, which is responsible (Sareyan et al. 1997, Handler et al. 2006).

- Understand the spectral window of your data. See Sect. 1.

- Pre-whiten. See Sect. 2.

- Quit before you get too close to the level of noise. See Breger et al. (1999).

- Compare the synthetic light (or velocity) curve with the data. If you don’t, you risk an unpleasant surprise. An example is shown in Fig. 3.

- When you compare the frequencies you derived with earlier work, look for differences close to 1.003 and 0.003 c/d. The first number is approximately equal to 1 c/sd (see Sect. 1), the second is close to 1 cycle per year (c/y). An alias of 1 c/y (or a fraction thereof, such as 1/2, 1/3, etc.) is much more difficult to get rid of than a 1 c/sd alias.

An opinion

If the data are OK, any method of analysis will do, provided that the method is used properly.

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DISCUSSION

Kovács: What is your opinion about using pure (truncated) Gaussian noise on the observed time base in a Monte Carlo simulations to get an estimate of the noise level?

Jerzykiewicz: It is a useful exercise which may serve as a guide. However, statistical properties of real noise are seldom, if ever, known.